Vertex functions and generalized normal-ordering by triple systems in non-linear spinor field models*

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Triple systems are closely related to Yang-Baxter symmetries. Utilizing a non-parameter-dependent triple product, we derive the BCS interaction. The enlargement of the notion of symmetry leads in some sense to a regular vertex function. The connection to the effect of running coupling constants is outlined, which leads to the recently discussed anisotropic effective local interactions. Furthermore, a discussion of the physical nature of q-symmetries is given.

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I. INTRODUCTION

In a couple of papers, Susumu Okubo has shown the equivalence of triple products and the Yang-Baxter symmetry [2–4]. These triple systems [5–7] provide an easy way to find solutions of the Yang-Baxter equation in several cases. Okubo distinguishes orthogonal triple systems (OTS) and symplectic triple systems (STS) due to the symmetry of the involved bilinear forms.

In this short note, we want to emphasize a new direction of the application of this method. The main idea is developed from the property of triple products to map $\mathcal{V}\otimes\mathcal{V}\otimes\mathcal{V}\mapsto\mathcal{V}$. Since an anharmonic interaction is of such a cubic structure, we are able to utilize this structure for a simplification. The same argument holds for the vertex function of non-linear spinor fields.

From a historic point, triple systems are related to non-associative, alternative algebras and especially Jordan algebras. These algebras were developed in the same time as the von Neumann quantum mechanics [8,9] to avoid the complex numbers [10,11]. This was expressed in the name r-number contrary to c-numbers or q-numbers of the new mechanics. Already Jordan proofed, that the Jordan algebras are equally well suited to handle QM as the operator algebras are, except from some exceptional cases sic. the exceptional Jordan algebras. Due to the difficulties in calculating with non-associative algebras, Jordan algebras were not accepted in physics beside some special cases.

So they became useful in supersymmetric string models and conformal field theory [12,13]. The Nambu mechanics [14], which is a generalization of Hamilton mechanics by multiple Hamiltonians leads to Jordan algebra structures. The quantum analog is related to para-fermions [15,16] as was shown by Yamaleev and others [17,18]. In this context, it is important to note, that Jordan algebras appear naturally as subalgebras of Clifford algebras [19], which are closely related to the spinor fields we want to employ in section III. Furthermore, we find a close connection between so called generalized Clifford algebras [20–22] and the Duffin-Kemmer-Petiau equations and matrices [23,24], as again to para-fermions. This motivates the attempt to employ triple systems and triple products to spinor modules.

The relation between triple systems and non-associative algebras is established via the notation of the associator, which can be defined in an algebra $\mathcal{A}$ with the binary, linear relations ($a,b,c,c'\in\mathcal{A}$)

$$\begin{aligned} + &: \mathcal{A}\times\mathcal{A}\mapsto\mathcal{A} \\ &\quad (a;b)\rightarrow a+b=b+a=c \\ m &: \mathcal{A}\times\mathcal{A}\mapsto\mathcal{A} \\ &\quad (a;b)\rightarrow m(a,b)=ab=c', \end{aligned} \tag{1.1}$$

subjected to the usual compatibility requirement of the distributive law

*This article is based on chapter two of the author's thesis[1]

$$m(a,b+c) = m(a,b) + m(a,c)$$
$$m(b+c,a) = m(b,a) + m(c,a) \tag{1.2}$$

as the ternary relation – associator:

$$(\,,\,,\,) : \mathcal{A} \times \mathcal{A} \times \mathcal{A} \mapsto \mathcal{A}$$
$$(a;b;c) \to (a,b,c) = (ab)c - a(bc). \tag{1.3}$$

The associator is thus a measurement of the deviation from associativity in an analogous sense as the commutator is a measure for the lack of commutativity.

Furthermore, we want to emphasize, that ternary algebraic structures are of recent interest for its own [25], because one may describe quark properties very elegant within them. Also one has to notice, that not every ternary multiplication relation can be expressed as an algebraic expression of the binary multiplication and addition [26], as we do in this paper.

The paper is organized as follows:

In section II we give the definition of OTS. Thereby we state in II A the defining relations in an axiomatic way, establish in II B the connection to the Yang-Baxter-equation and provide in II C a motivation of the defining relations. Their physical relevance is emphasized.

Section III is devoted to the application of a particular triple system and a discussion of its generalization and physical implications. A vertex function of a non-linear spinor field model is obtained due to strict application of triple product methods in III A, which in the particular case results in the BCS dynamics when restricted to the low energy – non-relativistic – regime.

Subsection III B describes a generalization of the results in III A and connects the transition from non-θ-dependent triple systems to θ-dependent ones with a generalized ordering of interacting quantum theories. The connection to normal-ordering – for the special case discussed in III A – as the relevance to recent developements is given.

Subsection III C speculates about an alternative interpretation of q-deformed variables i.e. the quantum (hyper) plane and q-symmetry. The results from III A and III B motivate and support the new approach to look at q-symmetry as the symmetry of composite structures and *not* as a fundamental physical issue.

The paper closes with a summary in section IV.

II. ORTHOGONAL TRIPLE SYSTEMS

In this section we give an outline of the mathematical structure of triple systems. We follow closely Okubo [2–4]. Because of the aim to apply these techniques to non-linear spinor fields, most emphasis is laid in this preparatory section on a physical motivation of triple systems.

A. Definition of orthogonal triple systems

Let $\mathcal{V}$ be a finite dimensional $\mathbb{K}$-vector space of dimension N over $\mathbb{K}$. We denote elements of $\mathcal{V}$ with small latin letters. Let $\mathcal{V}$ be equipped with a non-degenerate bilinear form $<.|.>$ – scalar product –, hence with $x, y \in \mathcal{V}$ $\alpha \in \mathbb{K}$

$$<.|.> \;: \mathcal{V} \times \mathcal{V} \mapsto \mathbb{K}$$
$$<x|y> = \alpha$$
$$<x|y> = 0 \quad \forall y \Rightarrow x \equiv 0. \tag{2.1}$$

Denote the triple product as $[x,y,z]$ with range in $\mathcal{V}$. An orthogonal triple system is defined by the following conditions, where $x,y,z,u,v,w \in \mathcal{V}$ and $\alpha, \beta \in \mathbb{K}$

$$
\begin{aligned}
i)&\quad <x|y> = <y|x> \\
ii)&\quad [x,y,z] = [y,z,x] = -[z,y,x] \\
iii)&\quad <w|[x,y,z]> \quad \text{antisymmetric in } x,y,z \text{ and } w \\
iv)&\quad <[x,y,z]|[u,v,w]> = \alpha\Big\{ \\
&\quad <x|u><y|v><z|w> + <y|u><z|v><x|w> \\
&\quad + <z|u><x|v><y|w> - <x|w><y|v><z|u>
\end{aligned}
$$

$$
\begin{aligned}
&- < y|w >< z|v >< w|u > - < z|w >< x|v >< y|v > \Big\} \\
&+\beta\Big\{ < x|u >< y|[z,v,w] > + < y|u >< z|[x,v,w] > \\
&+ < z|u >< x|[y,v,w] > + < x|v >< y|[z,w,u] > \\
&+ < y|v >< z|[x,w,u] > + < z|v >< x|[y,w,u] > \\
&+ < x|w >< y|[z,u,v] > + < y|w >< z|[x,u,v] > \\
&+ < z|w >< x|[y,u,v] > \Big\}.
\end{aligned} \tag{2.2}
$$

One may equivalently write equation (2.2-iv) in the form

$$
\begin{aligned}
iv)' \quad & [[x,y,z],u,v] = \Big\{ \\
& \alpha\{< y|v >< z|u > - < y|u >< z|v >\} - \beta < u|[v,y,z] > \Big\}x \\
& +\alpha\{< z|v >< x|u > - < z|u >< x|v >\} - \beta < u|[v,z,x] > \Big\}y \\
& +\alpha\{< x|v >< y|u > - < x|u >< y|v >\} - \beta < u|[v,x,y] > \Big\}z \\
& -\beta\Big\{ < x|v > [u,y,z] + < y|v > [u,z,x] + < z|v > [u,x,y] \\
& + < x|u > [v,z,y] + < y|u > [v,x,z] + < z|u > [v,y,x]\Big\}.
\end{aligned} \tag{2.3}
$$

This triple system is general enough for the specific application in section III. But, if we would like to consider physical systems where the coupling constant is not constant but a function of some parameter θ, as e.g. the energy, we have to introduce a θ-dependent triple product. Of special interest is the QCD, where one has the phenomenon of a so-called "running coupling constant", see discussions in subsections III B and III C below.

The θ-dependent triple system is defined via the θ-dependent triple product – note the positions of variables in LHS and RHS –

$$
\begin{aligned}
[z,x,y]_\theta := & P(\theta)[x,y,z] \\
& +A(\theta) < x|y > z + B(\theta) < z|x > y + C(\theta) < z|y > x,
\end{aligned} \tag{2.4}
$$

where $A(\theta), B(\theta), C(\theta), P(\theta)$ are functions of the parameter, see Okubo [3,4].

B. Relation of orthogonal triple systems to the Yang-Baxter equation

The idea to construct regular vertex functions of non-linear spinor field models with the help of triple systems arose from the following connection of orthogonal triple systems and the Yang-Baxter–equation (YB), which is known to be a sort of "integrability condition". This "regularization" will be used in section III. Now we turn to the formulation of the Yang-Baxter equation within *special* triple systems. Let $\{e_a\}$ be an arbitrary base of $\mathcal{V}$ and define a dual base $\{e^b\}$ with help of the scalar product by

$$
< e^b|e_a > \; := \delta^b_a. \tag{2.5}
$$

Strictly spoken, e^b is an *image* of a covector in the vector space $\mathcal{V}$, since we couldn't write down a scalar product otherwise. Because of the non-degeneracy of the scalar product, we have an isomorphism between $\mathcal{V}^*$ and $\mathcal{V}$ which allows such a writing.

Now define the Yang-Baxter–matrix $R^{ab}_{cd}(\theta)$ as

$$
R^{ab}_{cd}(\theta)e_a = [e^b, e_c, e_d]_\theta. \tag{2.6}
$$

One is now able to write the Yang-Baxter equation alternatively in matrix or triple product form [3]. The equivalent expressions are given with $\theta' = \theta + \theta''$ as either

$$
R^{b'a'}_{a_1a_2}(\theta)R^{c'a_2}_{cd}(\theta')R^{c_2b_2}_{b'c'}(\theta'') = R^{c'b'}_{b_1c_1}(\theta'')R^{c_2a'}_{a_1c'}(\theta')R^{b_2a_2}_{a'b'}(\theta) \tag{2.7}
$$

or

$$
[v,[u,e_j,z]_{\theta'},[e^j,x,y]_\theta]_{\theta''} = [u,[v,e_j,x]_{\theta'},[e^j,z,y]_{\theta''}]_\theta, \tag{2.8}
$$

where we have used the abbreviations $x = e_{a_1}$, $y = e_{b_1}$, $z = e_{c_1}$, $u = e^{a_2}$ and $v = e^{c_2}$. Usually θ is called the "rapidity parameter".

Now it is well known from the theory of integrable systems, that solutions of this equation behave regular. Hence, a triple system which additionally satisfy equation (2.8) provides one with a solution of the Yang-Baxter–equation and is supposed to provide a good candidate for a vertex function in a non-linear spinor field model.

We will see in the next paragraph that triple systems are very restricted. Nevertheless, it is sufficient for a first application to study only non-θ-dependent triple systems, which are of course *not* solutions of the YB-equation. A discussion of the generalization will be given in the section III B.

C. Motivation of the triple system relations

To give a further motivation of the triple system relation, we assume L to be a semisimple (simple) Lie–algebra and $\mathcal{V}$ to be an irreducible L-module – representation space with irreducible representation.

Any tensor product of such representation spaces can be decomposed due to the symmetric group. Hence we can decompose $\mathcal{V} \otimes \mathcal{V}$ into symmetric $(\mathcal{V} \otimes \mathcal{V})_S$ and antisymmetric $(\mathcal{V} \otimes \mathcal{V})_A$ parts as

$$(\mathcal{V} \otimes \mathcal{V}) := (\mathcal{V} \otimes \mathcal{V})_S + (\mathcal{V} \otimes \mathcal{V})_A. \tag{2.9}$$

For higher tensor products one may use Young tableaux to characterize the appropriate decomposition [27]. This tableaux can be equivalently described by partitions, e.g. $[1^3]$ for three totally antisymmetric spaces, $[3]$ for totally symmetric and $[2,1]$ for a mixed symmetry.

If the L-module $\mathcal{V}$ $(\mathcal{V} = [1])$ fulfills the conditions

$$\begin{aligned} i)&\ \ Dim\ Hom((\mathcal{V} \otimes \mathcal{V})_S \to \mathbf{K}) = 1 \\ ii)&\ \ Dim\ Hom([1^3] \to \mathcal{V}) = 1 \\ iii)&\ \ Dim\ Hom([1^4] \to \mathbf{K}) = 1 \\ iv)&\ \ Dim\ Hom(([1^3] \otimes [1^3])_S \to \mathbf{K}) \leq 2, \end{aligned} \tag{2.10}$$

where $Dim\mathcal{W}$ is the dimension of the space of homomorphisms $\mathcal{W} = Hom(\mathcal{W}_1 \to \mathcal{W}_2)$ of L-modules from $\mathcal{W}_1$ into $\mathcal{W}_2$ which are compatible with the action of L, then (2.10) implys the relations of the triple system (2.2).

(2.10-i) provides one the unique existence of a symmetric inner product. Because of the invariance of $\mathcal{V}$ under the action of L, this inner product is non-degenerate. From this we can conclude that there exists a unique isomorphism from $\mathcal{V}^*$ onto $\mathcal{V}$ which allows the identification of $\mathcal{V}^*$ and $\mathcal{V}$. Because we have restricted ourselves to orthogonal triple systems, one can conclude from eq. (2.2-i) the following symmetry of the YB–R–matrix:

$$R^{ab}_{cd}(\theta) = R^{ba}_{dc}(\theta). \tag{2.11}$$

This is an additional symmetry, but symplectic cases are also possible.

The existence of one and only one total antisymmetric triple product is an outcome of the condition (2.10-ii). This condition restricts the dimension of $\mathcal{V}$.

The restriction (2.10-iii) yields the possibility to build a unique scalar product between an element from $\mathcal{V}$ and a triple product.

For the interpretation of (2.10-iv) we notice, that triple systems are closely related to algebras with composition law. Let

$$\begin{aligned} \times\ :\ &\mathcal{V} \times \mathcal{V} \mapsto \mathcal{V} \\ &(x; y) \to x \times y = z, \end{aligned} \tag{2.12}$$

then composition means the validity of the following identity

$$< x \times y | x \times y > \ = < x | x >< y | y > . \tag{2.13}$$

This is very close to the well known formula of determinants

$$det(AB) = det(A)det(B), \tag{2.14}$$

see [28]. The connection between (2.2-iv) and (2.10-iv) follows from this structure. First, we have to notice, that the only (finitely generated) division ($\mathbf{R}$-)algebras are $\mathbf{R}$ itself, $\mathbf{C}$ the complex numbers, $\mathbf{H}$ the Hamilton quaternions and $\mathbf{O}$ the Cayley octonions, which are of dimension 1,2,4 and 8, see [28].

To establish the connection, we choose an arbitrary element e, with the properties

$$< e|e >= 1 \tag{2.15}$$

and

$$e \times x = x \times e = x, \tag{2.16}$$

with the above defined binary algebra product. We use the symbol $\times$ to emphasize the close connection to cross-product algebras, which can be found in dimension 3 and 7 due to the reduction by singling out the "real" element e. With the associator and the commutator defined as

$$\begin{aligned} i)\;\; & (x, y, z) := (x \times y) \times z - x \times (y \times z) \\ ii)\;\; & [x, y] := x \times y - y \times x, \end{aligned} \tag{2.17}$$

it is possible to express the triple product now as

$$[x, y, z] := \frac{1}{2}\Big\{(x, y, z)+ < x|e > [y, z]+ < y|e > [z, x]+ < z|e > [x, y]- < z|[x, y] > e\Big\}. \tag{2.18}$$

To show the connection to a quaternion algebra (octonion algebra in 8 dim. case) one can introduce a conjugation

$$\bar{x} := 2 < x|e > e - x, \tag{2.19}$$

which is obviously an involution. Identifying e with the unit in **H**, $e \equiv 1$ and introducing an orthogonal normed base $\{i, j, k\}$ in the "imaginary" part of **H** yields the analogy. For more complicated triple systems and higher dimensions see [3].

III. APPLICATION TO A NON-LINEAR SPINOR FILED

A. "Regular vertex" from a triple system

Since we are mainly interested in the formulation of a non-linear spinor field model, we omit the parameter dependent coupling constant. We explicitly submit that this is an artificially restriction. One should also note the more advanced spinor equations of Daviau and Lochak [29–31], which are much more general.

Hence we look for spinor theories with a four fermion interaction like

$$g V_I^{I_1 I_2 I_3} \psi_{I_1} \psi_{I_2} \psi_{I_3}, \tag{3.1}$$

where g is a real constant, $V_I^{I_1 I_2 I_3}$ is a constant vertex and $\psi_{I_1} \psi_{I_2} \psi_{I_3}$ are spinor fields. This kind of model is found in various very distinct physical areas, as in solid state theory e.g. the Hubbard model [32] or in the hadronization of QCD as effective low energy limit, where we would have an Index I, which also includes the color degrees of freedom, e.g. [33]. The important case is, however, the θ-dependent generalization, which will be discussed in the next subsection.

We omit also the general problems of defining quantized non-linear spinor field theories, which are discussed elsewhere [1], by assuming a sufficient regularization e.g. by a cutoff method.

A further problem, arising when quantizing non-linear theories is ordering. To "quantize" a theory means to give a heuristic concept how to translate a given classical theory into *one* corresponding quantum theory. This is possible in general for free theories only, where "p" and "q"–variables occur only quadratic, see the Groenwald–van Howe theorem discussed in [34]. The ordering problem was treated in [35,1] for fermionic fields and will hereafter not be dealt with. We emphasize, that a treatment which includes ordering could be given if the current development would be treated in the algebraic picture promoted in [1].

To be able to determine *a particular* vertex function, we have to search for solutions of the non-θ-dependent triple system (2.2). Therefore we cite the following

theorem: (Okubo [3]) There are three classes of solutions of the triple system (2.2):

$$\begin{aligned} & a)\; N = 8, \quad \text{with } \alpha^2 = \beta \\ & b)\; N = 4, \quad \text{with } \beta = 0 \\ & c)\; [[x, y, z], u, v] \equiv 0, \quad \alpha = \beta = 0. \end{aligned} \tag{3.2}$$

Since c) is trivial and a) leads to L-modules of dimension 8 and the Cayley octonions, we restrict ourselves to the case b), simply to exemplify our idea.

Hence we have now $Dim\mathcal{V} = N = 4$, and we may choose an orthonormal base, elements of that kind are called e_μ where $\mu \in \{1,2,3,4\}$. Furthermore we obtain in this simple case

$$< e_\mu | e_\nu > \; = \delta_{\mu\nu}. \tag{3.3}$$

This enables us to write (2.2-iii) as

$$< e_\alpha | [e_\mu, e_\nu, e_\lambda] > \; = \epsilon_{\alpha\mu\nu\lambda}, \tag{3.4}$$

where ϵ is the fully antisymmetric pseudo-tensor of Levi–Civita. Now, we proceed by distinguishing an element e, which has to obtain the relations $< e|e >= 1$, $e \cdot x = x \cdot e = x$, where the binary dot-multiplication is defined to be

$$\begin{aligned} \cdot\, : \mathcal{V} \times \mathcal{V} &\mapsto \mathcal{V} \\ (x; y) &\to x \cdot y = z. \end{aligned} \tag{3.5}$$

Hence, singling out a special element, which provides us also with an special involution etc., we can connect the triple product and a bilinear product via associator (2.17-i) and commutator (2.17-ii) due to (2.18) by

$$\begin{aligned} [x, y, z] = \frac{1}{2}\Big\{ &(x, y, z) + < x|e > [y, z] + < y|e > [z, x] \\ &+ < z|e > [x, y] - < z|[x, y] > e \Big\}. \end{aligned} \tag{3.6}$$

We are very close to quaternions here, consider the conjugation (2.19) we may set $e \equiv 1$ and find an orthogonal base of the "imaginary" part of $\mathbf{H}$ – the Hamilton quaterion algebra – $\{i, j, k\}$. One obtains then

$$\begin{aligned} x \cdot y = &[x, y, e] \\ &+ < x|e > y + < y|e > x - < x|y > e, \end{aligned} \tag{3.7}$$

which leads for "imaginary" – or say traceless – elements of $\mathcal{V}$ to the equations

$$x \cdot y + y \cdot x = -2 < x|y > e \tag{3.8}$$

$$x \cdot y - y \cdot x = 2[x, y, e], \tag{3.9}$$

which shows the Clifford algebraic nature of the dot-multiplication, as the close connection of cross-products and triple products, which can be found in the 7 dimensional "imaginary" part of the octonion algebra also [28].

An element composed of a real part and an higher dimensional imaginary "vector" part is usually called a para-vector [36,37]. This concept depends on the grading, where the imaginary part here is based on the kernel of *the* linear form induced by the special element e via, $\lambda(x) :=< x|e >$, $\lambda(e) = 1$. Since we have developed the above theory over the complex field, this analogy is not strict, but could be made so. Nevertheless, we would have to change the involved concept of gradation.

Now we are prepared to solve our problem for the purpose of a non-linear spinor field theory. We look for (complex) spinors of four dimensions, hence elements of a linear space – spinor space – of four dimensions. This spinor space has in a natural manner an L-modul structure due to the action of the γ-matrices. The corresponding metric is non degenerate, symmetric and the spinor representation is irreducible. Hence we have fulfilled the requirements of the triple system (2.2) and may apply the above theorem and obtain its special solution, by setting

$$V_\alpha^{\alpha_1\alpha_2\alpha_3} := \epsilon_{\alpha\alpha_1\alpha_2\alpha_3}, \tag{3.10}$$

here without color or isospin degrees of freedom. If we develop the ϵ-tensor into tensor products of γ matrices, we obtain in the symmetric base, which relies on spinor – charge conjugated spinor spaces the following expression, using summation convention (s.c.)

$$\epsilon_{\alpha\alpha_1\alpha_2\alpha_3} = \frac{1}{2}\Big\{ \gamma^\mu\gamma^5 C \otimes \gamma_\mu\gamma^5 C + C \otimes C - \gamma^5 C \otimes \gamma^5 C \Big\}_{\alpha\alpha_1\alpha_2\alpha_3}. \tag{3.11}$$

If Dirac-representation is used, one has $C = i\gamma^2\gamma^0$. In field theoretic language we have an axial vector, a scalar and a pseudo scalar part in this expression. If we would like to have a Fierz-symmetric form, we may use appendix D of [38] to obtain (s.c.)

$$
\begin{aligned}
\epsilon_{\alpha\alpha_1\alpha_2\alpha_3} = \Big\{ & -\frac{1}{4}C\otimes C + \frac{3}{4}\gamma^5 C\otimes\gamma^5 C \\
& -\frac{1}{8}\gamma^\mu C\otimes\gamma_\mu C - \frac{1}{4}\gamma^\mu\gamma^5 C\otimes\gamma_\mu\gamma^5 C + \frac{1}{8}\Sigma_{\mu\nu}C\otimes\Sigma^{\mu\nu}C\Big\}_{\alpha\alpha_1\alpha_2\alpha_3}.
\end{aligned} \tag{3.12}
$$

In this form, we obtain *all* cases of possible couplings, but with a *fixed relative* strength. Hence we are able to formulate our "vertex regularized" non-linear spinor theory, which reads quantized

$$
\begin{aligned}
(i\gamma^\mu\partial_\mu - m_i)^{reg}_{\alpha\beta}\Psi_\beta(x) &= g\epsilon_{\alpha\alpha_1\alpha_2\alpha_3}\Psi_{\alpha_1}\Psi_{\alpha_2}\Psi_{\alpha_3}(x) \\
\{\Psi_{\Lambda_1}(t,\vec{\mathbf{r}}_1), \Psi^\dagger_{\Lambda_2}(t,\vec{\mathbf{r}}_2)\}_+ &= \delta_{\Lambda_1\Lambda_2}\delta(\vec{\mathbf{r}}_1-\vec{\mathbf{r}}_2),
\end{aligned} \tag{3.13}
$$

where *reg* means cutoff or something else. We have used since now a condensed notation, where ψ_α is a spinor composed of a pair of spinors ψ_Λ, $\psi^\dagger_\Lambda$ or respectively the charge conjugated spinor.

If we look now at the low energy non-relativistic limit, and omit also self energy of mass, we obtain

$$
-\frac{\Delta}{2m}\psi_\alpha = g\epsilon_{\alpha\alpha_1\alpha_2\alpha_3}\psi_{\alpha_1}\psi_{\alpha_2}\psi_{\alpha_3}(x). \tag{3.14}
$$

This is the equation of motion of electrons in a BCS-superconductor after the phonon–electron interaction has been removed in favor of the local electron–electron interaction [39].

The "regularity" of this type of vertex function was studied first time in [35]. A fully quantum filed theoretic treatment, based on generating functionals was given in [1]. There and in [40,41] we found a connection between regularity and the representation and gradation of the theory. This will be important in the next subsection. However, we emphasize, that due to the kinetic term, the theory is, without say a cutoff, still singular.

B. θ-dependence and ordering

Since we have restricted ourself to the simple non-θ-dependent triple system (2.2), we want to give some further aspects of the general case in this subsection.

The relation of θ-dependent and non-θ-dependent triple systems was given in (2.4), where the scalar functions $A(\theta), B(\theta), C(\theta), P(\theta)$ of the "rapidity parameter" θ were defined also. This triple system is general and *not* subjected to the YB-equation, which can be seen as an *additional* constraint due to (2.8) of the triple system.

Now, we found in [35] the following relation for elements of the appropriate L-modul

$$
a\,\dot\wedge\, b\,\dot\wedge\, c := a\wedge b\wedge c + F_{ab}c + F_{bc}a + F_{ca}b, \tag{3.15}
$$

which was obtained by a normal-ordering procedure. This does reduce in QFT Fock space to the usual Wick-Dyson normal-ordering [35]. The wedge $\wedge$ and dotted-wedge $\dot\wedge$ are two different representations of the exterior algebra onto the considered Clifford algebra – sic. quantized exterior algebra –, which induce there different gradings [35]. Recall, that normal-ordering is performed as the lowest step of the transition of QFT generating functionals to connected ones. Hence, a normal-ordering extracts the two-particle correlation and leads to an two-particle irreducible generating functional. This technique is well known from statistical mechanics [42].

Since the above defined systems are antisymmetric and noticing (2.2-iii), we may write

$$
[x,y,z] \equiv x\wedge y\wedge z \tag{3.16}
$$

for the non-θ-dependent triple product and

$$
[z,x,y]_\theta \equiv x\,\dot\wedge\, y\,\dot\wedge\, z \tag{3.17}
$$

for the – here antisymmetric – θ-dependent triple product. If we would define now

$$
\begin{aligned}
P(\theta) &\equiv 1 \\
A(\theta) < x|y > &\equiv F_{xy} \\
B(\theta) < z|x > &\equiv F_{zx} \\
C(\theta) < z|y > &\equiv F_{zy},
\end{aligned} \tag{3.18}
$$

we would arrive at the same equation as for the normal-ordering (3.15) which was found in [35]. The YB-equation would thus lead to a further requirement which has to be fulfilled by the functions F_{ab}. In [43,1] we have shown, that

F is nothing but the *propagator* of the full, i.e. interacting, theory. It was also shown, that F determines the proper grading of the involved algebras and hence such notations as "particle number, scalar, vector, tensor, ..." etc., see also [40,41].

Looking for the solutions of the YB-equation, found by Okubo for the above special case, we arrive at

$$\begin{aligned}\frac{A(\theta)}{P(\theta)} &= -a \\ \frac{B(\theta)}{P(\theta)} &= a + b\theta \\ \frac{C(\theta)}{P(\theta)} &= a + \frac{a^2-\alpha}{b\theta}\end{aligned} \tag{3.19}$$

and hence at

$$[z,x,y]_\theta = P(\theta)\Big\{[x,y,z] - a < x|y > z + (a+b\theta) < z|x > y + (a + \frac{a^2-\alpha}{b\theta}) < z|y > x\Big\}. \tag{3.20}$$

We obtain thus, up to the overall factor $P(\theta)$, a generalized "normal-ordering" procedure due to the YB-symmetry. The propagator function F_{xy} is no longer isotrop, but depends on the position of the factors which are *contracted*. Hence, with the above ordering of variables we have

$$\begin{aligned}F^1_{xy} &= -a < x|y > \\ F^2_{zx} &= (a+b\theta) < z|x > \\ F^3_{zy} &= (a + \frac{a^2-\alpha}{b\theta}) < z|y > .\end{aligned} \tag{3.21}$$

We may arrive at the usual isotropic ordering procedure up to the overall factor $P(\theta)|_{\theta=-2a/b}$ by letting simultaneous

$$\begin{aligned}b\theta &\rightarrow -2a \\ \alpha &\rightarrow \ 5a^2,\end{aligned} \tag{3.22}$$

which reduces the three different non-isotropic propagators to $F_{st} = -a < s|t >$ and hence to

$$[z,x,y]_{-2a/b} = P(\frac{-2a}{b})\Big\{[x,y,z] + F_{xy}z + F_{zx}y + F_{zy}x\Big\}. \tag{3.23}$$

This is up to the overall P-factor equivalent to (3.15) and proofs thereby the existence of the appropriate limit. Defining in general (in our case)

$$\begin{aligned}< e^a|[e^b, e_c, e_d] > &= C^{cd}_{ab} & (\cong \epsilon^{cd}_{ab}) \\ < e^a|e^b > &= g^{ab} & (\cong \delta^{ab}) \\ < e_a|e_b > &= g_{ab} = (g^{ab})^{-1} & (\cong \delta_{ab}),\end{aligned} \tag{3.24}$$

we arrive at the following relation for the YB-R-matrix

$$R^{ab}_{cd}(\theta) = P(\theta)C^{ab}_{cd} + A(\theta)g_{cd}g^{ab} + B(\theta)\delta^a_d\delta^b_c + C(\theta)\delta^a_c\delta^b_d \tag{3.25}$$

respectively

$$\begin{aligned}R^{ab}_{cd}(\theta) &= P(\theta)\Big\{\epsilon^{ab}_{cd} - a\delta_{cd}\delta^{ab} + (a+b\theta)\delta^a_d\delta^b_c + (a + \frac{a^2-\alpha}{b\theta})\delta^a_c\delta^b_d\Big\} \\ &(\cong P(\frac{-2a}{b})\Big\{\epsilon^{ab}_{cd} - a\delta_{cd}\delta^{ab} - a\delta^a_d\delta^b_c - a\delta^a_c\delta^b_d\Big\}).\end{aligned} \tag{3.26}$$

The overall factor $P(\theta)$ would in our spinor model lead to a θ-dependent coupling constant $g(\theta) = gP(\theta)$. But indeed this is a formal analogy and further investigations on this topic are necessary.

The importance of the obtained result is not founded in the special model, but in a novel and unexpected relation to recent effective models in QCD [44]. These authors considered (randomly distributed) anisotropic color-interactions discussing confinement in an effective non-linear spinor field model, essentially equivalent in structure to our model.

A further important application of such models lays in high-T_c-super conductor models, which can be described by analogous methods. Both models, in high energy and solid state physics, rely basically on the so called Anderson localization [45], which was the starting point for the model considered in [44] and ultimately lead to the anisotropic interaction there.

In contrast to these considerations, our approach was motivated from an algebraic setting, which was modeled to obtain somehow distinguished – regular – interactions which are related to the generalized q-symmetry. Some consequences of this results will be discussed in the next section.

C. On the nature of quantum deformations

Since we have connected in a totally different way the YB-symmetry and hence therewith deformed symmetries with physical theories, we may try to give some alternative comments on the physical meaning of q-deformed symmetries and variables. For some expository texts see [46].

Usually, one can develop from the quantum plane $A_q^{p|0}$ and the dual $A_q^{0|p}$ the theory of deformations [47]. One is then confronted with the intriguing concept of an *non-commutative point space*. There are very much speculations on the physical meaning of such constructions. Most of this considerations do connect the discrete structure, which comes along with q-deformation, with the physics at the Planck scale [48]. Since we are interested in low energy solid state physics or the low energy behavior of QCD, we cannot account for the Planck scale and related ideas of a discrete space-time [49]. Additionally, one arrives at q-deformed structures by studying differential calculi on Lattices [50]. This lead already to speculations, that a discrete space-time may be found underlying conventional continuous space-time. Such considerations may be the origin of the extensive study of deformed Lorentz and Poicare symmetry [51]. The difficult point is, that the deformation parameters, if there are many, are *not* subjected to the dynamics of the system. Deformation provides the background of a theory, but not the details. From QFT it is clear, that the dynamics of a system *has to choose* a particular – non-Fock, if the theory is not free – representation. The representation is in general subjected to the dynamics, which forbids in general a fixed, non-dynamical, deformation.

Our treatment of the four fermion local interaction suggests another attempt to explain this structure. The most important fact is the usability of the generalized symmetry. This is the foundation of the integrability of systems which satisfy the YB-equation. If we reject the quantum plane as a *point space*, we remain with all benefits of the involved symmetry.

We state therefore, that the q-variables model *some* aspects of composites in multi-particle systems. Non-commutativ spaces therefore does not conflict with the usual physical point space. The preparation of composites has to be conserved *somehow* under the dynamics. This cannot be an ordinary isometry, as long as the composite is not entirely rigid and the composite density is low – weak interaction of composites assumed.

Our simple consideration above supplies this point of view. The local electron-electron interaction arose due to the elimination of the intermediate phonon interaction. The same situation is considered in effective QCD theories [44]. Hence, we may look at the spinors ψ_α as composed of an electron (quark) and somehow phonons (gluons), which could be described by higher spin-tensors. The YB-equation is then on the one hand an integrability condition and on the other the condition, how to be able to map the general structure of composites onto another.

We may state, that q-deformed symmetry is the symmetry of composites – entities with dynamical internal structure. The q-parameters are parameters of this structures, which define the possible types of "congruent" structures, and are therefore not dynamic. Due to the connection to the usually employed normal-ordering of QFT, we obtain a direct connection between q-parameters and the *state space* of the theory [1]. The connection of the YB-solutions and non-isotropic contractions – sic. propagators – was outlined above, see also [35,40,1,41]. Since contractions rely on two-particle issues, this is a direct way to see the multi-particle character of q-symmetry. An extensive discussion of the connection of quantum mechanical state space and q-symmetry will be given elsewhere.

IV. CONCLUSION

We developed, by applying triple systems, a method to obtain a "regular vertex" in a non-linear spinor field model. However, the regularity treated in [35,1] was not made explicit. The main idea was to show the usefulness of triple products in the determination of distinguished vertex functions. The easy and elegant handling of YB-symmetry by special triple systems was used to give a distinguished vertex, which proved to be in the non-relativistic case the BCS interaction, when phonons are eliminated in favor of a local electron-electron interaction. The assumptions underlying the employed triple system were physically motivated. The requirements are in a non-mathematical language, that first one assumes a non-degenerate symmetric scalar product. Second, one forces the triple product to behave as

an *effective single field*. Third, there is a scalar product between single fields and triples. Fourth, there is a scalar product on these effective fields.

All in all, these assumptions guarantee the theory to be considered as an effective theory. This physical background remains true if θ-dependent triple systems are considered.

Furthermore, we gave an outline, how this procedure can be generalized to the important class of models with parameter dependent – running – coupling constants. Only in this case one obtains a YB-symmetry.

The recent interest in anisotropic local four fermion interactions gives direct relevance to our investigations. Every solution of the YB-equation involved in the θ-dependent triple system leads, up to a θ-dependent overall factor, to a fixed anisotropic vertex function and constrains the physical possibilities drastically.

The θ-dependent triple system also lead to a generalized *non-isotropic ordering procedure* which was shown to be equivalent to the usual Wick-Dyson normal-ordering in the non-θ-dependent limit.

The new approach to deformed symmetries was used to give an alternative interpretation of q-symmetries and q-variables. We rejected q-variables as *point space*, but looked at them as a sort of effective coordinates of composites which exhibit non-trivial internal dynamics. The investigation with help of triple systems and their relation to generalized non-isotropic orderings does support this point of view.

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